Matrix Transformations Using Eigenvectors

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Outline

- Review last lecture
- Transformations with a matrix of eigenvectors: $\Lambda = \mathbf{X}^{-1}\mathbf{AX}$
- Hermitian and orthogonal matrices
- Quadratic forms
- Numerical methods for eigenvalues and eigenvectors

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Review Eigens

- Basic definition (**A** n x n)**:** $\mathbf{Ax} = \lambda\mathbf{x}$
- Det $[\mathbf{A} - \mathbf{I}\lambda] = 0$ gives n^{th} order equation for eigenvalues
  - n eigenvalues (may not be distinct)
  - solve $[\mathbf{A} - \mathbf{I}\lambda]\mathbf{x} = \mathbf{0}$ for n components of each of n eigenvectors
  - eigenvectors undetermined to within a multiplicative constant
  - eigenvectors may or may not be linearly independent

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Transform **A** into a Diagonal

- If the n eigenvectors of **A** are linearly independent we can define an invertible matrix, **X**, whose columns are eigen-vectors of **A**: $\mathbf{X} = [\mathbf{x}_{(1)}\ \mathbf{x}_{(2)}\ \mathbf{x}_{(3)}\ \ldots\ \mathbf{x}_{(n)}]$
- $\mathbf{AX} = [\mathbf{Ax}_{(1)}\ \mathbf{Ax}_{(2)}\ \mathbf{Ax}_{(3)}\ \ldots\ \mathbf{Ax}_{(n)}]$
- $\mathbf{AX} = [\lambda_1\mathbf{x}_{(1)}\ \lambda_2\mathbf{x}_{(2)}\ \lambda_3\mathbf{x}_{(3)}\ \ldots\ \lambda_n\mathbf{x}_{(n)}]$
- We now show that $\mathbf{AX} = \Lambda\mathbf{D}$ where Λ is a diagonal matrix of eigenvalues

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Matrix Product $\mathbf{X}\Lambda$

$$\mathbf{X}\Lambda = \begin{bmatrix} x_{(1)1} & x_{(2)1} & x_{(3)1} & \cdots & \cdots & x_{(n)1} \\ x_{(1)2} & x_{(2)2} & x_{(3)2} & \cdots & \cdots & x_{(n)2} \\ x_{(1)3} & x_{(2)3} & x_{(3)3} & \cdots & \cdots & x_{(n)3} \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ x_{(1)n} & x_{(2)n} & x_{(3)n} & \cdots & \cdots & x_{(n)n} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & \lambda_n \end{bmatrix}$$

- Usual **X** matrix component is, $x_{row,column}$
- This **X** component notation is $x_{(vector)row}$
- Usual matrix multiplication formula applies

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Matrix Product $\mathbf{X}\Lambda$ Continued

$$\mathbf{X}\Lambda = \begin{bmatrix} \lambda_1 x_{(1)1} & \lambda_2 x_{(2)1} & \lambda_3 x_{(3)1} & \cdots & \cdots & \lambda_n x_{(n)1} \\ \lambda_1 x_{(1)2} & \lambda_2 x_{(2)2} & \lambda_3 x_{(3)2} & \cdots & \cdots & \lambda_n x_{(n)2} \\ \lambda_1 x_{(1)3} & \lambda_2 x_{(2)3} & \lambda_3 x_{(3)3} & \cdots & \cdots & \lambda_n x_{(n)3} \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ \lambda_1 x_{(1)n} & \lambda_2 x_{(2)n} & \lambda_3 x_{(3)n} & \cdots & \cdots & \lambda_n x_{(n)n} \end{bmatrix}$$

$$= \left[\lambda_1\mathbf{x}_{(1)}\ \ \lambda_1\mathbf{x}_{(2)}\ \ \lambda_1\mathbf{x}_{(3)}\ \ \cdots\ \cdots\ \ \lambda_1\mathbf{x}_{(n)}\right] = \mathbf{AX}$$

- $\mathbf{AX} = [\lambda_1\mathbf{x}_{(1)}\ \lambda_2\mathbf{x}_{(2)}\ \lambda_3\mathbf{x}_{(3)}\ \ldots\ \lambda_n\mathbf{x}_{(n)}]$ from previous slide. We now see that $\mathbf{AX} = \mathbf{X}\Lambda$

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A Transformed

- We assumed that **X** has an inverse; we can pre-multiply **AX** = **X**Λ by this inverse to get

$$\mathbf{X}^{-1}\mathbf{A}\mathbf{X} = \mathbf{X}^{-1}\mathbf{X}\Lambda = \Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & \lambda_n \end{bmatrix}$$

$$\boxed{\Lambda = \mathbf{X}^{-1}\mathbf{A}\mathbf{X}}$$

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Transform Example for 2 x 2

- Last class example: eigenvectors for 2 x 2 matrix **A**, with $\lambda_1 = 2$ and $\lambda_2 = 1$

$$\mathbf{A} = \begin{bmatrix} 1 & 5 \\ 0 & 2 \end{bmatrix} \qquad \mathbf{x}_{(1)} = \begin{bmatrix} 5\alpha \\ \alpha \end{bmatrix} \qquad \mathbf{x}_{(2)} = \begin{bmatrix} \beta \\ 0 \end{bmatrix}$$

- The **X** matrix and its inverse are

$$\mathbf{X} = \begin{bmatrix} 5\alpha & \beta \\ \alpha & 0 \end{bmatrix} \qquad \mathbf{X}^{-1} = \frac{1}{(5\alpha)(0)-(\alpha)(\beta)}\begin{bmatrix} 0 & -\beta \\ -\alpha & 5\alpha \end{bmatrix}$$

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Check Inverse, Comupte **AX**

$$\mathbf{X}^{-1} = -\frac{1}{\alpha\beta}\begin{bmatrix} 0 & -\beta \\ -\alpha & 5\alpha \end{bmatrix} \qquad \textit{Does } \mathbf{X}^{-1}\mathbf{X} = \mathbf{I}\,?$$

$$\mathbf{X}^{-1}\mathbf{X} = -\frac{1}{\alpha\beta}\begin{bmatrix} 0 & -\beta \\ -\alpha & 5\alpha \end{bmatrix}\begin{bmatrix} 5\alpha & \beta \\ \alpha & 0 \end{bmatrix}$$

$$= -\frac{1}{\alpha\beta}\begin{bmatrix} (0)(5\alpha)+(-\beta)(\alpha) & (0)(\beta)+(-\beta)(0) \\ (-\alpha)(5\alpha)+(5\alpha)(\alpha) & (-\alpha)(\beta)+(5\alpha)(0) \end{bmatrix} = \mathbf{I}$$

$$\mathbf{A}\mathbf{X} = \begin{bmatrix} 1 & 5 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 5\alpha & \beta \\ \alpha & 0 \end{bmatrix} = \begin{bmatrix} (1)(5\alpha)+(5)(\alpha) & (1)(\beta)+(5)(0) \\ (0)(5\alpha)+(2)(\alpha) & (0)(\beta)+(2)(0) \end{bmatrix}$$

$$= \begin{bmatrix} 10\alpha & \beta \\ 2\alpha & 0 \end{bmatrix}$$

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Transform Example Result

$$\mathbf{X}^{-1}\mathbf{A}\mathbf{X} = -\frac{1}{\alpha\beta}\begin{bmatrix} 0 & -\beta \\ -\alpha & 5\alpha \end{bmatrix}\begin{bmatrix} 10\alpha & \beta \\ 2\alpha & 0 \end{bmatrix} =$$

$$-\frac{1}{\alpha\beta}\begin{bmatrix} (0)(10\alpha)+(-\beta)(2\alpha) & (0)(\beta)+(-\beta)(0) \\ (-\alpha)(10\alpha)+(5\alpha)(2\alpha) & (-\alpha)(\beta)+(5\alpha)(0) \end{bmatrix}$$

$$= -\frac{1}{\alpha\beta}\begin{bmatrix} -2\alpha\beta & 0 \\ 0 & -\alpha\beta \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

- This example produces the expected result: $\mathbf{X}^{-1}\mathbf{A}\mathbf{X}$ is a diagonal matrix of eigenvalues (regardless of α and β)

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Another $\mathbf{X}^{-1}\mathbf{A}\mathbf{X}$ Example

- Last class 3 x 3 example had **A** matrix with eigenvalues $\lambda_1 = -2$, $\lambda_2 = -2$, $\lambda_3 = 6$ and eigenvectors shown below

$$\mathbf{A} = \begin{bmatrix} 2 & 2 & -6 \\ 2 & -1 & -3 \\ -2 & -1 & 1 \end{bmatrix} \quad \mathbf{x}_{(1)} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \quad \mathbf{x}_{(2)} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \quad \mathbf{x}_{(3)} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 2 & 0 & 1 \end{bmatrix} \quad \mathbf{X}^{-1} = \begin{bmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 2 & 0 & 1 \end{bmatrix}^{-1} = \frac{1}{16}\begin{bmatrix} 2 & 1 & 5 \\ -2 & 7 & 3 \\ -4 & -2 & 6 \end{bmatrix}$$

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Another $\mathbf{X}^{-1}\mathbf{A}\mathbf{X}$ Example II

- Using matrices $\mathbf{X}^{-1}$ and **A** from the previous slide we have

$$\mathbf{X}^{-1}\mathbf{A} = \frac{1}{16}\begin{bmatrix} 2 & 1 & 5 \\ -2 & 7 & 3 \\ -4 & -2 & 6 \end{bmatrix}\begin{bmatrix} 2 & 2 & -6 \\ 2 & -1 & -3 \\ -2 & -1 & 1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -2 & -1 & -5 \\ 2 & -7 & -3 \\ -12 & -6 & 18 \end{bmatrix}$$

$$\mathbf{X}^{-1}\mathbf{A}\mathbf{X} = \frac{1}{8}\begin{bmatrix} -2 & -1 & -5 \\ 2 & -7 & -3 \\ -12 & -6 & 18 \end{bmatrix}\begin{bmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

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Orthogonal Matrices

- An orthogonal matrix has mutually orthogonal columns, $\mathbf{a}_{(j)}$
- Write matrix as $[\mathbf{a}_{(1)} \ \mathbf{a}_{(2)} \ \mathbf{a}_{(3)} \ \dots \ \mathbf{a}_{(n)}]$
- $(\mathbf{a}_{(i)}, \mathbf{a}_{(j)}) = \mathbf{a}^T_{(i)}\mathbf{a}_{(j)} = \Sigma a_{ki}a_{kj} = \delta_{ij}$
- Summation formula is equivalent to matrix multiplication of $\mathbf{A}^T\mathbf{A} = \mathbf{I}$
- Thus, $\mathbf{A}^T = \mathbf{A}^{-1}$ for orthogonal matrices
- Both rows and columns are orthogonal

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More on Orthogonal Matrices

- A vector transform with an orthogonal matrix preserves the vector length
- For $\mathbf{y} = \mathbf{Ax}$, with $\mathbf{A}$ orthogonal, $||\mathbf{y}||^2 = \mathbf{y}^T\mathbf{y} = (\mathbf{Ax})^T\mathbf{Ax} = \mathbf{x}^T\mathbf{A}^T\mathbf{Ax}$
- Orthogonal matrix: $\mathbf{A}^T = \mathbf{A}^{-1}$ so $\mathbf{A}^T\mathbf{A} = \mathbf{I}$
- So, $||\mathbf{y}||^2 = \mathbf{x}^T\mathbf{A}^T\mathbf{Ax} = \mathbf{x}^T\mathbf{Ix} = \mathbf{x}^T\mathbf{x} = ||\mathbf{x}||^2$
- Conclusion: when $\mathbf{y} = \mathbf{Ax}$, with $\mathbf{A}$ orthogonal, $||\mathbf{y}||^2 = ||\mathbf{x}||^2$

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Hermitian/Symmetric Matrices

- Symmetric matrix: $\mathbf{A} = \mathbf{A}^T$
- Hermitian matrix: $\mathbf{A}^H = \mathbf{A}^\dagger = (\mathbf{A}^*)^T = \mathbf{A}$
- A real symmetric matrix is a Hermitian matrix (also called self-adjoint)
- For an n x n Hermitian matrix
  - Eigenvalues are real
  - Eigenvectors form a linearly independent, orthogonal basis set in n dimensions
    - May have complex eigenvectors for complex **A**

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Unitary Matrix

- Analog of an orthogonal matrix for complex-values matrices
- For a unitary matrix, $\mathbf{U}$, $\mathbf{U}^H = \mathbf{U}^{-1}$
  - I. e. for a unitary matrix we get the inverse by taking the transpose and setting all values of i to –i
- Eigenvectors of a Hermitian matrix, **A**
  - Form an orthogonal matrix for real-valued **A** and a unitary matrix if **A** has complex values

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Hermitian Eigenvectors

- Recall **X** matrix whose columns are eigenvectors giving $\Lambda = \mathbf{X}^{-1}\mathbf{AX}$
- Requires **X** to have inverse
- This is guaranteed for a Hermitian **A**
- Furthermore, since **X** columns are orthogonal eigenvectors, $\mathbf{X}^{-1} = \mathbf{X}^H$, which is the same as $\mathbf{X}^T$ for real **A**
- Example of real Hermitian matrix, **A**

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 1 \end{bmatrix}$$

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Hermitian Example

- Solve Det[$\mathbf{A} - \mathbf{I}\lambda$] = 0 for eigenvalues
- $\lambda_1 = 4.7131967$, $\lambda_2 = -2.789263462$, and $\lambda_3 = 0.076066756$
- Solve $(\mathbf{A} - \mathbf{I}\lambda_k)\mathbf{x}_{(k)} = \mathbf{0}$ for unit eigenvectors and construct **X** matrix

$$\mathbf{X} = \begin{bmatrix} 0.494321 & 0.270183 & -0.826225 \\ 0.606278 & -0.788297 & 0.104950 \\ 0.622955 & 0.552801 & 0.553478 \end{bmatrix}$$

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Hermitian Example Continued

- Show eigenvectors are orthonormal

$$\mathbf{X} = \begin{bmatrix} 0.494321 & 0.270183 & -0.826225 \\ 0.606278 & -0.788297 & 0.104950 \\ 0.622955 & 0.552801 & 0.553478 \end{bmatrix}$$

$$\left(\mathbf{x}_{(1)}, \mathbf{x}_{(2)}\right) = \mathbf{x}_{(1)}^T \mathbf{x}_{(2)} = (0.494321)(0.270183) + (0.606278)(-0.788297) + (0.622955)(0.552801) = 0$$

- Can show $(\mathbf{x}_{(i)}, \mathbf{x}_{(j)}) = \delta_{ij}$

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Hermitian Example Concluded

- Since columns of **X** are orthogonal, **X** is an orthogonal matrix: $\mathbf{X}^{-1} = \mathbf{X}^T$

$$\mathbf{X}^{-1} = \begin{bmatrix} 0.494321 & 0.606278 & 0.622955 \\ 0.270183 & -0.788297 & 0.552801 \\ -0.826225 & 0.104950 & 0.553478 \end{bmatrix}$$

- Can verify this by taking inverse
- Can also show that $\mathbf{X}^{-1}\mathbf{A}\mathbf{X} = \text{diag}[\lambda_1, \lambda_2, \lambda_3]$

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Another Hermitian Example

- Find **X** such that $\mathbf{X}^{-1}\mathbf{A}\mathbf{X} = \Lambda$ for $\mathbf{A} = \mathbf{A}^H$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad Det(\mathbf{A} - \mathbf{I}\lambda) = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$Det(\mathbf{A} - \mathbf{I}\lambda) = (1-\lambda)^3 + 0 + 0 - (1-\lambda) - 0 - 0$$
$$= (1-\lambda)\left[(1-\lambda)^2 - 1\right] = (1-\lambda)(1 - 2\lambda + \lambda^2 - 1)$$
$$= (1-\lambda)\lambda(-2+\lambda) = 0$$

$\lambda = 2, 1, 0$

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Another Hermitian Example II

- Now that eigenvalues are known find eigen-vectors from $[\mathbf{A} - \mathbf{I}\lambda] = \mathbf{0}$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1-\lambda_k & 0 & 1 \\ 0 & 1-\lambda_k & 0 \\ 1 & 0 & 1-\lambda_k \end{bmatrix} \begin{bmatrix} x_{(k)1} \\ x_{(k)2} \\ x_{(k)3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Get eigenvector components for each of the eigenvalues

For $\lambda_1 = 2$, $1 - \lambda_1 = -1$

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Another Hermitian Example III

- Apply general equation to $\lambda_1 = 2$

$$\begin{bmatrix} 1-\lambda_k & 0 & 1 \\ 0 & 1-\lambda_k & 0 \\ 1 & 0 & 1-\lambda_k \end{bmatrix} \begin{bmatrix} x_{(k)1} \\ x_{(k)2} \\ x_{(k)3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_{(1)1} \\ x_{(1)2} \\ x_{(1)3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{(1)1} \\ x_{(1)2} \\ x_{(1)3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Solution is $x_{(1)3} = a$, $x_{(1)2} = 0$, and $x_{(1)1} = a$, for any value of a

For $\lambda_1 = 2$, $1 - \lambda_1 = -1$

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Another Hermitian Example IV

- Apply general equation to $\lambda_2 = 1$

$$\begin{bmatrix} 1-\lambda_k & 0 & 1 \\ 0 & 1-\lambda_k & 0 \\ 1 & 0 & 1-\lambda_k \end{bmatrix} \begin{bmatrix} x_{(k)1} \\ x_{(k)2} \\ x_{(k)3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{(2)1} \\ x_{(2)2} \\ x_{(2)3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Solution is $x_{(2)3} = 0$, $x_{(2)2} = b$, and $x_{(2)1} = 0$, for any value of b

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Another Hermitian Example V

- Apply general equation to $\lambda_3 = 0$ $\begin{bmatrix} 1-\lambda_k & 0 & 1 \\ 0 & 1-\lambda_k & 0 \\ 1 & 0 & 1-\lambda_k \end{bmatrix} \begin{bmatrix} x_{(k)1} \\ x_{(k)2} \\ x_{(k)3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{(3)1} \\ x_{(3)2} \\ x_{(3)3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{(3)1} \\ x_{(3)2} \\ x_{(3)3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Solution is $x_{(3)3} = c$, $x_{(3)2} = 0$, and $x_{(3)1} = -c$, for any value of c

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Another Hermitian Example VI

- General result $\mathbf{x}_{(1)} = \begin{bmatrix} a \\ 0 \\ a \end{bmatrix}$ $\mathbf{x}_{(2)} = \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix}$ $\mathbf{x}_{(3)} = \begin{bmatrix} -c \\ 0 \\ c \end{bmatrix}$
- Set a = b = c = 1 $\mathbf{x}_{(1)} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\mathbf{x}_{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\mathbf{x}_{(3)} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$
- Normalized vectors $\mathbf{x}_{(1)} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$ $\mathbf{x}_{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\mathbf{x}_{(3)} = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$

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Another Hermitian Example VII

$$(\mathbf{x}_{(1)}, \mathbf{x}_{(2)}) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0 + 0 + 0 = 0$$

$$(\mathbf{x}_{(2)}, \mathbf{x}_{(3)}) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = 0 + 0 + 0 = 0$$

$$(\mathbf{x}_{(1)}, \mathbf{x}_{(3)}) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = -1 + 0 + 1 = 0$$

- Inner (dot) products of unlike vectors are zero for orthogonal set

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Another Hermitian Example VIII

- Form **X** matrix from individual eigenvectors $\mathbf{x}_{(1)} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$ $\mathbf{x}_{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\mathbf{x}_{(3)} = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$
- $\mathbf{X}^{-1}$ should equal $\mathbf{X}^T$ for orthogonal eigenvector matrix

$$\mathbf{X} = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

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Another Hermitian Example IX

- Check to see if $\mathbf{X}^{-1} = \mathbf{X}^T$(true if $\mathbf{X}\mathbf{X}^T = \mathbf{I}$

$$\mathbf{X}\mathbf{X}^T = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$\mathbf{X}\mathbf{X}^T = \begin{bmatrix} 1/2 + 0 + 1/2 & 0+0+0 & 1/2 + 0 - 1/2 \\ 0/\sqrt{2} + 0 - 0/\sqrt{2} & 0+1+0 & 0+0+0 \\ 1/2 + 0 - 1/2 & 0+0+0 & 1/2 + 0 + 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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Another Hermitian Example X

$$\mathbf{X}^{-1}\mathbf{A}\mathbf{X} = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \bullet$$

Details on next slide

$$\begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{\Lambda}$$

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Another Hermitian Example XI

$$\mathbf{X}^{-1}\mathbf{A} = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{2} & 0 & 2/\sqrt{2} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{X}^{-1}\mathbf{A}\mathbf{X} = \begin{bmatrix} 2/\sqrt{2} & 0 & 2/\sqrt{2} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{\Lambda}$$

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Quadratic Form

- Q = $\mathbf{x}^T\mathbf{A}\mathbf{x}$, with symmetric **A**
- Q = $\mathbf{x}^H\mathbf{A}\mathbf{x}$ with Hermitian **A**
- Both the same if **A** has all real values

$$\mathbf{x}^H\mathbf{A}\mathbf{x} = \begin{bmatrix} x_1^* & x_2^* & x_3^* & \cdots & \cdots & x_n^* \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}$$

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Quadratic Form II

$$\mathbf{x}^H\mathbf{A}\mathbf{x} = \begin{bmatrix} x_1^* & x_2^* & x_3^* & \cdots & \cdots & x_n^* \end{bmatrix} \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots\cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots\cdots + a_{2n}x_n \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots\cdots + a_{3n}x_n \\ \vdots \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots\cdots + a_{nn}x_n \end{bmatrix}$$

$$\mathbf{x}^H\mathbf{A}\mathbf{x} = \begin{array}{l} x_1^*(a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots\cdots + a_{1n}x_n) + \\ x_2^*(a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots\cdots + a_{2n}x_n) + \\ x_3^*(a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots\cdots + a_{3n}x_n) + \\ \cdots \\ \cdots \\ x_n^*(a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots\cdots + a_{nn}x_n) \end{array}$$

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Quadratic Form III

$$\mathbf{x}^H\mathbf{A}\mathbf{x} = \sum_{i=1}^{n} x_i^* \sum_{j=1}^{n} a_{ij}x_j = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i^*x_j =$$

$$\sum_{i=1}^{n} a_{ii}x_i^2 + \sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} a_{ij}x_i^*x_j = \sum_{i=1}^{n}\left(a_{ii}x_i^2 + 2\sum_{j=i+1}^{n} a_{ij}x_i^*x_j\right)$$

- Positive definite if $\mathbf{x}^H\mathbf{A}\mathbf{x} \geq 0$ for any **x**
- Positive semidefinite if $\mathbf{x}^H\mathbf{A}\mathbf{x} > 0$ for any **x**
- Hermitian matrix is positive definite (semidefinite) if all its eigenvalues are positive (or zero)

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Similarity Transformations

- Transformations important in matrix operations and numerical analysis
- In the similarity transform $\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$, **B** will have the same eigenvalues as **A**
- $\mathbf{B}\mathbf{x} = \lambda\mathbf{x}$ => $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{x} = \lambda\mathbf{x}$
- Premultiply by **P** to get $\mathbf{P}\mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{x} = \mathbf{P}\lambda\mathbf{x}$
- $\mathbf{P}\mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{x} = \mathbf{I}\mathbf{A}\mathbf{P}\mathbf{x} = \mathbf{A}\mathbf{P}\mathbf{x} = \mathbf{P}\lambda\mathbf{x} = \lambda\mathbf{P}\mathbf{x}$
- **A** eigenvectors are **Px**

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Numerical Eigenvalue/vector

- Based on similarity transformations
- Householder/Givens transformations convert matrix to diagonal form
- Use library programs such as LINPAK, Visual Numerics IMSL library or MATLAB for calculations
- Can get range for eigenvalues by inclusion theorems

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Gerschgorin Inclusion Theorem

- Provides a set of (usually) overlapping disks on the complex plane that contain the eigenvalues

$$\left|\lambda - a_{jj}\right| \le \sum_{k=1}^{j-1}\left|a_{jk}\right| + \sum_{k=j+1}^{n}\left|a_{jk}\right| = \sum_{k=1,\,k\neq j}^{n}\left|a_{jk}\right|$$

- Apply to each diagonal element to get a disk with center (a_{jj}) and a radius $|\lambda - a_{jj}|$ in complex plane by row sum

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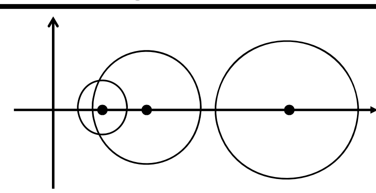
Gerschgorin Disk Example



- All eigenvalues lie in disks constructed from equation on previous chart
- Hermitian matrix eigenvalues must lie along real axis

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